Partitions in topological spaces and guessing principles

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Outline

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- Surfaces in the literature prominently during the 70s;

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- Partitions in Topological spaces (Coloring and topological structure of monochromatic subsets).

Definition

Let X and Y be topological spaces. The expression

$$X \to (Y)^1_\kappa$$

means that, for all $f \in \kappa^X$, there is an $\eta \in \kappa$ and a subset $Z \subset f^{-1}[\{\eta\}]$, such that Z is homeomorphic to Y.

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When considering ordinals with the order topology we specify the notation, writing: $X \to (top\,\alpha)^1_\kappa$.

Theorem ([2], Theorem 1)

Let X be a regular topological space with $X \to (top\,\omega + 1)^1_\omega$, *and* $\chi(X) < \mathfrak{b}$. *Then* $X \to (top\,\alpha)^1_\omega$ *for all* $\alpha < \omega_1$.

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What about the bound on the character in the theorem, is it the best possible?

Bound on the character

Theorem ([2], Theorem 4)

Suppose $\diamondsuit_{\omega_1}$ *holds. Then there exists a topological space* (X, τ) *such that:* $\chi(X) = \mathfrak{b}$, $(X, \tau) \to (\omega + 1)^1_\omega$, *and* $(X, \tau) \nrightarrow (\omega^2 + 1)^1_\omega$.

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The idea behind this result is to refine the neighborhoods of limit points in accordance with a diamond sequence that guesses colorings of ω_1 in ω many colors.This should suffice to destroy copies of $\omega^2 + 1$ while maintaining the monochromatic converging sequences.

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(2) *for all* $f : \omega_1 \to \omega$ *there are* $\alpha \in acc(\omega_1)$ *and* $n, m \in \omega$ *such that* $\alpha \in f^{-1}[\{n\}]$ *and* $A_\alpha^m \subset f^{-1}[\{n\}]$*.*

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Theorem

If $\clubsuit_F$ holds, then there is a regular topological space X such that $\chi(X) = \mathfrak{b}$, $X \to (top\,\omega + 1)^1_\omega$, but $X \not\to (top\,\omega^2 + 1)^1_\omega$.

Sketch of the proof:

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A Hajnal-Mátė graph is $G \subset [\omega_1]^2$ *such that the chromatic number of* G *is* $\aleph_1$ *and, for each* $\xi < \omega_1$, *the set* $\{\alpha < \xi : \{\alpha, \xi\} \in G\}$ *is either empty, or has order type* ω *and supremum* ξ.

We have the following result:

Proposition

If G is a Hajnal-Máté graph then there is a ladder system $\langle S_\alpha : \alpha \in \omega_1 \rangle$ such that, for each coloring $f : \omega_1 \to \omega$, there are $n \in \omega$ and $\alpha \in acc(\omega 1)$ such that S_α is cofinal in α of order type ω and $f(\alpha) = n$ and $f^{-1}[\{n\}] \cap S_\alpha$ is infinite.

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If there is a Hajnal-Maté graph, then there is a regular topological space X such that $\chi(X) = \mathfrak{b}$, $X \to (top\, \omega + 1)^1_\omega$, but $X \nrightarrow (top\, \omega^2 + 1)^1_\omega$.

Thank you for your time.

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